Some Theoretical Considerations Arising in Guidance Analysis

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Abstract

This paper discusses some of the theoretical considerations which arise when the guidance/navigation problem for space missions is treated as an application of optimal control and estimation theory. Expressions for the first and second variations of the trajectory end conditions are developed as functionals of the control variations, and necessary and sufficient conditions for optimality are described. The analogy to similar results in the ordinary calculus is emphasized. A geometrical interpretation of optimal control is presented, and the notions of controllability, abnormality, conjugate points, and extremal fields are discussed. The meaning of optimal control in the presence of random disturbances is discussed by introducing a simple problem from the ordinary calculus. A technique for sequentially estimating the time varying state vector (position and velocity) in the presence of noise on the navigation data is described. The resulting difference equations for the estimate and covariance matrix are extended to differential equations for the case of continuous data by postulating a "sequentially correlated" stochastic process.

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1. Navigation and Guidance for Space Missions

Navigation is the task of determining the state vector defining the spacecraft trajectory, where the state vector might be composed of the position and velocity components at some initial (epoch) time t_o, plus certain constant parameters which define the accelerations on the spacecraft subsequent to time t_o. Guidance is the task of calculating and executing corrective maneuvers which will cause the mission objectives to be achieved, using the navigation information to predict the motion of the spacecraft. (reference 1) The spacecraft trajectory can usually be approximated reasonably well by a series of "guidance phases," where the spacecraft is assumed to move under the influence of only one dominant gravitating body. In each such phase of flight various types and quality of navigation data are gathered, and various kinds of guidance corrections are accomplished. In figures 1, 2, and 3 the boost, earth escape, heliocentric transfer, approach, and terminal phases of the trajectory are illustrated for a typical interplanetary mission.

The data used to solve the navigation problem might consist of a series of celestial observations taken from the spacecraft, such as angles measured between certain stars and planets, and/or it might consist of earth-based radio tracking data, such as the radial speed of the spacecraft as measured by the doppler shift, and/or it might consist of the output of inertial measuring devices mounted on the spacecraft. The state vector can only be estimated, since the observed data would be contaminated with spurious noise, and/or may not contain sufficient information. The estimation procedure is usually designed to minimize the expected value of the squared error in the estimated, which is called minimum variance estimation. (reference 2)

The guidance corrections during a free fall phase of the trajectory would consist of one or more velocity impulses, imparted by a rocket engine which accelerates the spacecraft for a short period of time. The guidance corrections during a powered flight phase of the mission (while the vehicle is experiencing thrust acceleration) would consist of varying the direction of pointing the thrust vector, and/or of varying the thrust level, and/or of varying the time of terminating thrust. There are many ways to make corrections which will satisfy the mission objectives, but guidance is usually applied in such a way as to minimize (or maximize) some performance index. Typical would be minimizing the time required to accomplish the mission, or minimizing the required control effort. This approach to the guidance task gives rise to an optimal final value control problem or, equivalently, a problem in the calculus of variations. (references 3 and 4)

It is the purpose of this paper to discuss some of the theoretical considerations which arise when treating the navigation and guidance problem from the point of view of estimation and control theory. The ideas to be discussed will be illustrated by constructing simple examples which yield closed form solutions.

The notation employed is as follows: The independent variable is t, which may be thought of as time; T is the (fixed) final time; other capital letters are either matrices or kernels of integral equations; I is the identity matrix; column vectors are denoted by a bar (-) over a small letter; the transpose of a vector or matrix is indicated by the superscript '; δ refers to the variation of the indicated quantity from its standard value; and E[...] indicates the statistical expectation of the bracketed quantity. Partial derivatives will be written in compact matrix form, for example,

$$\left[\frac{\partial \beta_i}{\partial \bar{x}}\right]' = \left[\frac{\partial \beta_i}{\partial x_1}, \ \frac{\partial \beta_i}{\partial x_2} \ \ldots, \ \frac{\partial \beta_i}{\partial x_n}\right]$$

and $\left[\dfrac{\partial^2 \beta_i}{\partial x_j \partial x_k}\right]$ is a matrix with jk^{th} element equal to $\left(\dfrac{\partial^2 \beta_i}{\partial x_j \partial x_k}\right)$.
The (t) will occasionally be omitted in equations.

(insert figs 1, 2 and 3 here)

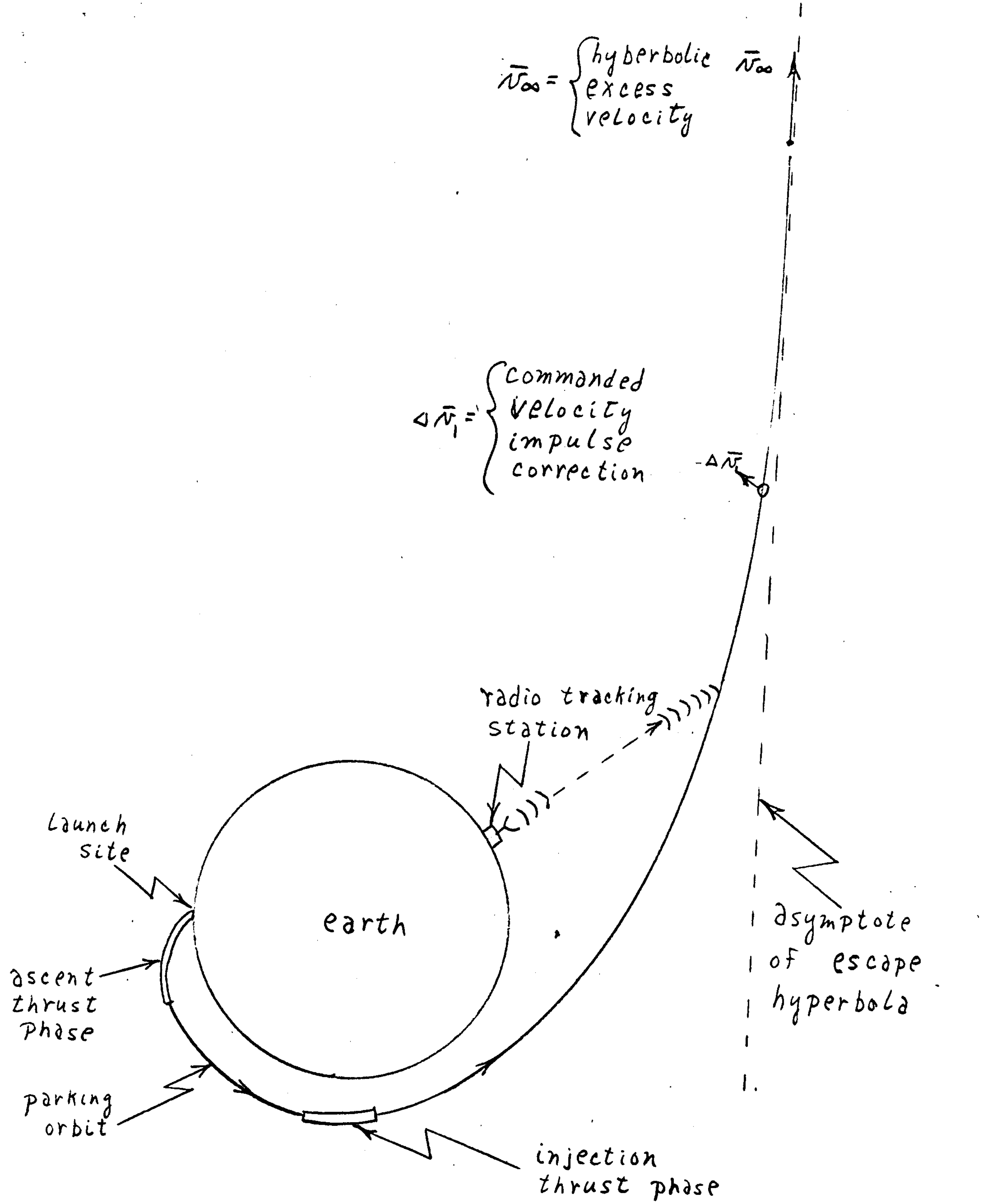

Fig 1: THE BOOST AND EARTH-ESCAPE PHASES OF AN INTERPLANETARY MISSION

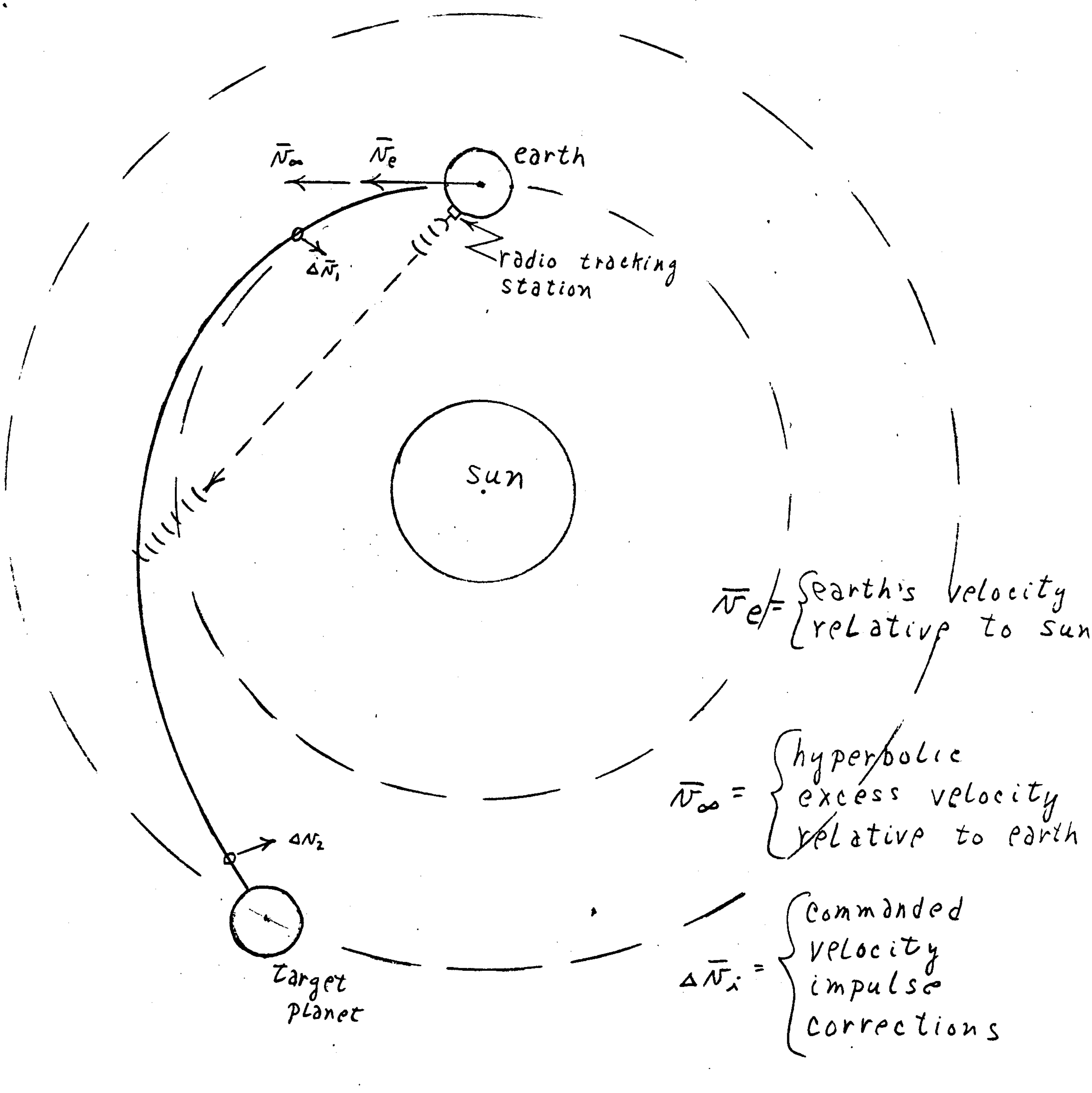

Fig 2: THE HELIOCENTRIC TRANSFER PHASE OF AN INTERPLANETARY MISSION

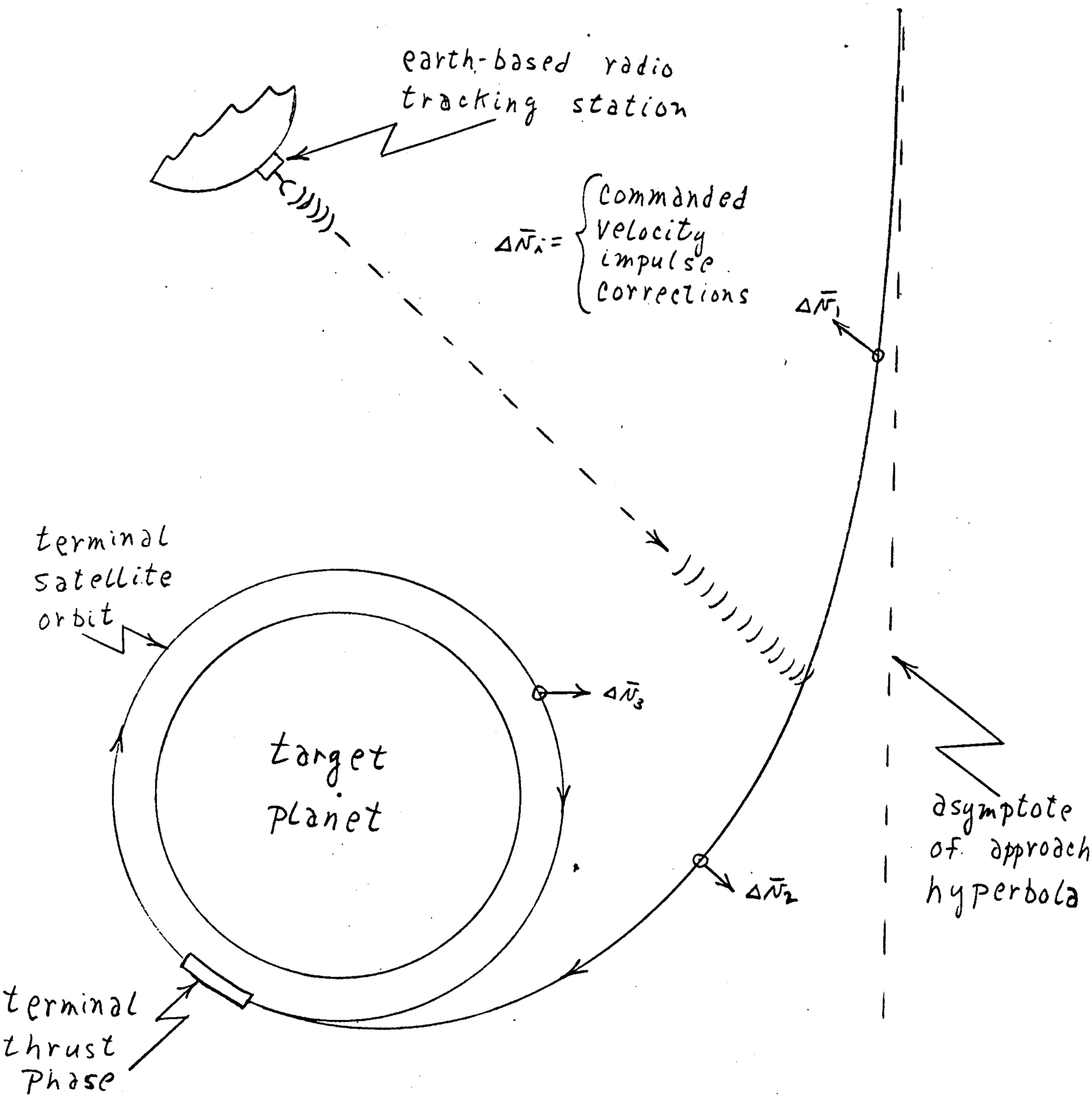

Fig 3: THE APPROACH AND TERMINAL PHASES OF AN INTERPLANETARY MISSION

2. Formulation of the Problem

The motion of a space probe is in general described by the vector differential equation[†]

$$\frac{d}{dt}\bar{x} = \bar{f}(\bar{x}, u, t) \tag{1}$$

where $\bar{x}$ is the six dimensional state vector composed of the position ($\bar{r}$) and velocity ($\bar{v}$) coordinates, u is some control variable, such as the steering angle or throttle setting of the rocket thrust vector, and t is the independent variable, usually taken to be time. (we shall consider here only the case of a single unbounded control variable) Thus equation (1) formulates the three components of the thrust and gravitational acceleration ($\frac{d}{dt}\bar{v}$) and the three components of the velocity ($\frac{d}{dt}\bar{r}$), and, for any given initial condition ($\bar{r}(t_1)$, $\bar{v}(t_1)$), the trajectory of the space probe can be determined if u(t) is specified.

For most applications equation (1) cannot be integrated in closed form, and numerical integration techniques are called for. To illustrate the ideas discussed here a simplified special example will be considered. We imagine the probe moving with constant speed on a unit sphere, and take the independent variable (t) to be longitude, which we assume to be always monotonically increasing. If x_1 is arc length, x_2 is latitude,

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[†]The situation where the forcing accelerations contain random elements will be discussed in Part 6.

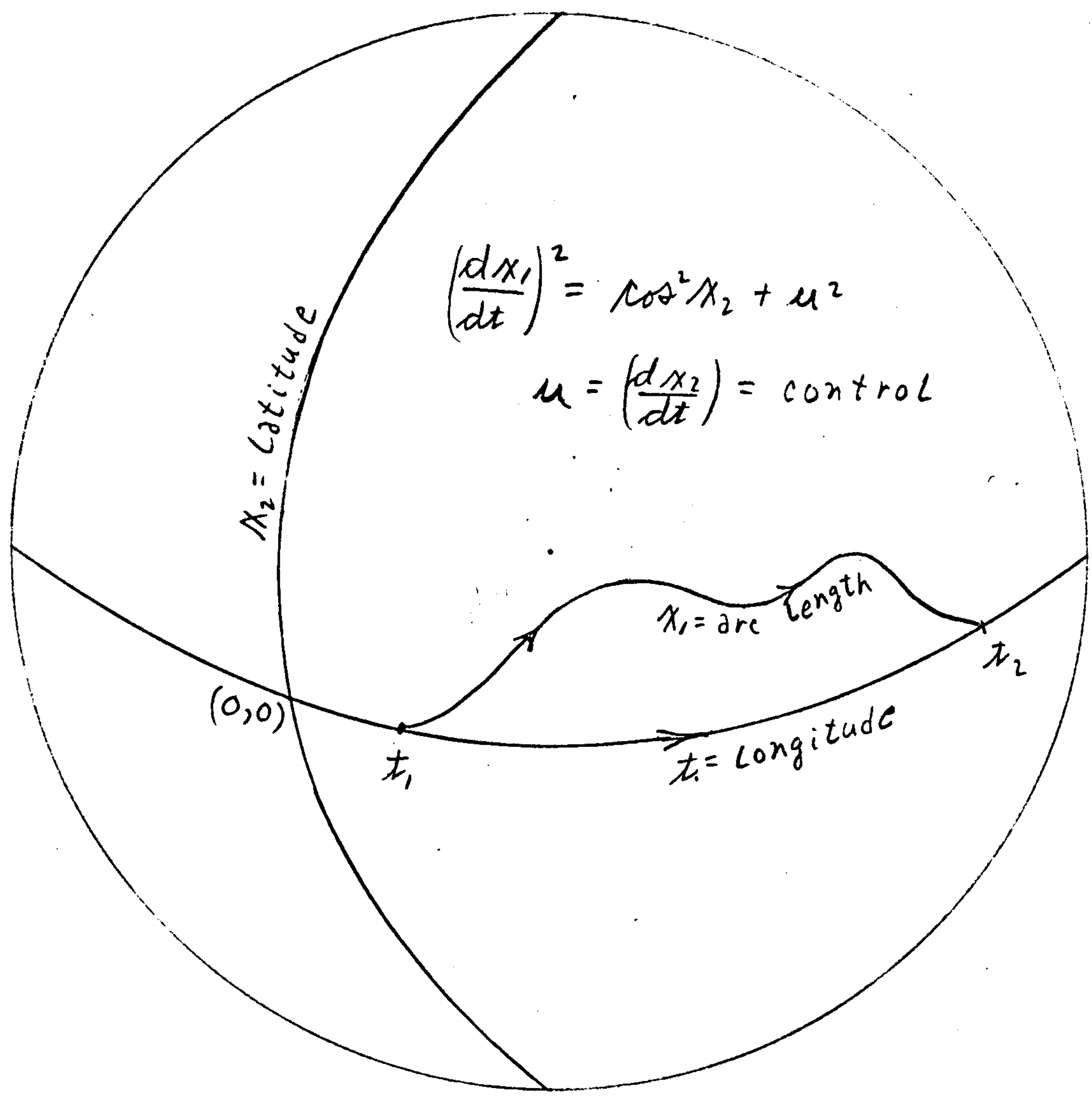

FIG. 4: MOTION ON THE UNIT SPHERE

and the control variable is the slope $u = \left(\frac{dx_2}{dt}\right)$, we have (see figure 4)

$$\frac{d}{dt}\bar{x} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} u^2 + \cos^2 x_2 \\ u \end{bmatrix} = \bar{f}(\bar{x}, u) \qquad (2)$$

represents time,

Note that the arc length x_1 ~~plays the role of time~~, since the time

traverse the longitude difference $(t_2 - t_1)$ at unit speed is simply $x_1(t_2) - x_1(t_1)$

required to ~~pass from t = 0 to t = T at speed v = 1 is simply x_1~~ This example can be interpreted as a crude model of motion around the sun from the earth to some other planet, where thrust acceleration is being continuously applied and the control variable u is related to steering angle of the thrust vector. The radius of the unit sphere would be the average radius of the earth and target planet.

(insert fig. 4 here)

3. The First and Second Variations

Perturbation theory is normally employed to formulate guidance and orbit determination equations, since it is usually not possible to find an explicit solution to the equations of motion (1). In this approach we postulate the existence of a known "standard", or reference, solution $\bar{x}_s(t)$ to equation (1), and consider the variations $\delta\bar{x}(t) = \bar{x}(t) - \bar{x}_s(t)$

developing a Taylor series expansion in

and $\delta\bar{u}(t) = \bar{u}(t) - \bar{u}_s(t)$. This approach is analogous to the ordinary calculus, where we are interested ~~to~~ in describing the behavior of some function† $\beta_i(\bar{u})$ in a small neighborhood of some given point $\bar{u}_s$. ~~described by~~ Thus

$$\delta\bar{\beta}_i = \beta_i(u) - \beta_i(u_s) = \left(\frac{\partial \beta_i}{\partial \bar{u}}\right)' \delta\bar{u} + \frac{1}{2}\delta\bar{u}'\left[\frac{\partial^2 \beta_i}{\partial u_i \partial u_j}\right]\delta\bar{u} + \text{higher order Terms} \tag{3}$$

where the partial derivatives are evaluated at $\bar{u} = \bar{u}_s$. If $|\delta\bar{u}|$ is sufficiently small, we may drop the higher order terms and consider the remaining second degree expression in the u_i. The term $\left(\left[\frac{\partial \beta_i}{\partial \bar{u}}\right]\delta\bar{u}\right)$ is called the first variation of the function $\beta_i(\bar{u})$ with respect to $\bar{u}$, and $\left(\delta\bar{u}'\left[\frac{\partial^2 \beta_i}{\partial u_i \partial u_j}\right]\delta\bar{u}\right)$ is called the second variation.

The situation is more complex when dealing with the functional $\beta_i\left(\bar{x}\{t_1, t_2; u(t)\}\right)$ where $\bar{x}$ is an n-dimensional state vector specified by the differential equation (1), and t_1, t_2 are (fixed) initial and final times. Thus β_i is implicitly a functional of the function u(t), instead of a function of the vector $\bar{u}$. If β_i is given as a linear combination of the x_i at t_2, that is

$$\beta_i\left(\bar{x}\{t_1, t_2; u(t)\}\right) = \sum_{j=1}^{n} a_j x_j\{t_1, t_2; u(t)\} \tag{4}$$

where $\bar{a}' = (a_1, a_2, \ldots a_n)$ is a constant vector, we have

$$\delta\beta_i = \bar{a}'\ \delta\bar{x}\{t_1, t_2; \delta u(t)\} \tag{5}$$

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†The subscript i is introduced because later we shall deal with more than one function $\beta_i(\bar{x})$.

We assume there is a standard trajectory described by $\bar{x}_s(t)$, $u_s(t)$, and employ equation (1) to obtain a Taylor series expansion for $\delta\dot{\bar{x}}$. ~~in the form~~ Thus

$$\frac{d}{dt}(\delta\bar{x}) = \delta\dot{\bar{x}} = F\delta\bar{x} + G\delta u + \tfrac{1}{2}H\delta u^2 + \tfrac{1}{2}\begin{bmatrix}\delta\bar{x}'J_1 \\ \vdots \\ \delta\bar{x}'J_n\end{bmatrix}\delta\bar{x} + M\delta\bar{x}\,\delta u + \text{higher order terms} \tag{6}$$

where the elements of the matrices F, G, H, J_k, and M are given by:

$$F_{ij} = \left(\frac{\partial f_i}{\partial x_j}\right) \quad (n \text{ by } n)$$

$$G_{i1} = \left(\frac{\partial f_i}{\partial u}\right) \quad (n \text{ by } 1)$$

$$H_{i1} = \left(\frac{\partial^2 f_i}{\partial u^2}\right) \quad (n \text{ by } 1)$$

$$(J_k)_{ij} = \left(\frac{\partial^2 f_k}{\partial x_i \partial x_j}\right) \quad (n \text{ by } n)$$

$$M_{ij} = \left(\frac{\partial^2 f_i}{\partial x_j \partial u}\right) \quad (n \text{ by } n)$$

and the partial derivatives are evaluated as functions of time along the standard trajectory. We define the n by n state transition matrix $U(t, \tau)$ by

$$\left.\begin{aligned} \frac{\partial U(t,\tau)}{\partial t} &= F(t)\, U(t,\tau) \\ \frac{\partial U(t,\tau)}{\partial \tau} &= -U(t,\tau)\, F(\tau) \\ U(t,t) &= \text{the identity} \end{aligned}\right\} \quad \text{for } t \geq \tau$$

$$U(t,\tau) = 0 \qquad \text{for } t < \tau$$

It follows that

$$\begin{aligned} \delta\beta_i\left(\bar{x}\{t_1, t_2; u(t)\}\right) = \bar{\lambda}'(t_1)\, \delta\bar{x}(t_1) &+ \int_{t_1}^{t_2} \eta_i(t)\, \delta u(t)\, dt \\ &+ \frac{1}{2} \int_{t_1}^{t_2} \xi_i(t)\, \delta u^2(t)\, dt \\ &+ \frac{1}{2} \int_{t_1}^{t_2}\!\!\int_{t_1}^{t_2} K_i(t,\tau)\, \delta u(t)\, \delta u(\tau)\, dt\, d\tau \\ &+ \text{higher order terms} \end{aligned} \tag{7}$$

where the "differential correction" vector is given by

$$\bar{\lambda}_i'(t) = \bar{a}'\, U(t_2, t) \tag{8}$$

the "impulse response function" is given by

$$\eta_i(t) = \bar{\lambda}_i'(t)\, G(t) \tag{9}$$

the "weighting function" is given by

$$\xi_i(t) = \bar{\lambda}_i'(t)\ H(t) \tag{10}$$

and the (symmetric) kernel is given by

$$K_i(t,\tau) = \lambda_i'(t)\ M_i(t)\ U(t,\tau)\ G(\tau) + \int_{t_1}^{t_2} \sum_{i=1}^{m} \lambda_i(s)\ G'(\tau)\ U'(s,\tau)\ J_i(s)\ U(s,t)\ G(t)\ ds \tag{11}$$

The sum of the first two terms on the right hand side of equation (7) is the first variation of β_i, and the sum of the second two terms, with the factor $\frac{1}{2}$ deleted, is the second variation.

This technique can be applied to our example problem if we postulate the standard trajectory to be the great circle arc $x_{2s}(t) = 0$, which results from the control $u_s(t) = 0$. The above described calculations are carried out in the appendix, where $\beta_o = x_1 \left\{ t_1, t_2; u(t) \right\}$ and $\beta_1 = x_2 \left\{ t_1, t_2; u(t) \right\}$. It is shown that

$$\delta\beta_o = \delta x_1(t_1) + \frac{1}{2}\int_{t_1}^{t_2} \delta u^2(t)\,dt + \frac{1}{2}\int_{t_1}^{t_2}\int_{t_1}^{t_2} K_o(t,\tau)\ \delta u(t)\ \delta u(\tau)\ dt\ d\tau + \text{higher order terms} \tag{12}$$

$$\delta\beta_1 = \delta x_2(t_1) + \int_{t_1}^{t_2} \delta u(t)\ dt + \text{higher order terms} \tag{13}$$

where

$$K_0(t,\tau) = \begin{cases} (t - t_2) & \text{for } t \geq \tau \\ (\tau - t_2) & \text{for } \tau > t \end{cases}$$

4. Optimal Final Value Control

The control function $u_s(t)$ is usually chosen to minimize (or maximize) some given performance index, such as the time required to accomplish the mission or the total control effort (cost). This becomes a problem in the classical calculus of variations (reference 4) or, equivalently, optimal control theory, and the well known Pontryagin Principle (reference 3) can be applied. Thus if $\beta_0\left(\bar{x}\{t_1, t_2; u(t)\}\right)$ is to be minimized, subject to $\beta_i\left(\bar{x}\{t_1, t_2; u(t)\}\right) = 0$ for i = 1, ...r, we define the "generalized hamiltonian" to be

$$h(\bar{x}, u, t) = \bar{\lambda}'(t)\, \bar{f}(\bar{x}, u, t) \tag{14}$$

where (see equation 8)

$$\bar{\lambda}(t) = \sum_{i=1}^{n} \nu_i\, \bar{\lambda}_i(t) \tag{15}$$

and the ν_i are the (constant) Lagrange multipliers. The optimal control must be chosen so that for all $\bar{x}$ and t we have

$$\left(\frac{\partial h}{\partial u}\right) = 0 \qquad \left(\frac{\partial^2 h}{\partial u^2}\right) \geq 0$$

We shall present another approach here, which follows from an analysis of the first and second variations. Just as in the ordinary calculus, it will be seen that necessary and sufficient conditions for optimality ~~can be~~ are obtained.‡

Let us consider first a simple problem in the ordinary calculus where we are to chose an m dimensional $\bar{u}_s$ such that

$$\begin{aligned} \beta_o &= \beta_o(\bar{u}_s) = \min \\ \beta_i &= \beta_i(\bar{u}_s) = 0 \quad \text{for } i = 1, \ldots r. < m \end{aligned} \tag{16}$$

To establish the first necessary condition we define the function

$$\beta_o^*(\bar{u}) = \sum_{i=0}^{r} \nu_i \, \beta_i(\bar{u}) = \bar{\nu}'\bar{\beta}(\bar{u}) \tag{17}$$

where ν_i are the Lagrange multipliers (which must be found by a search procedure), ~~and~~ chosen so that at the optimal $\bar{u}_s$ we have

$$(\bar{\eta}_o^*)' = \left(\frac{\partial \beta_o^*}{\partial \bar{u}}\right)' \triangleq \left[\left(\frac{\partial \beta_o^*}{\partial u_1}\right), \ldots\ldots, \left(\frac{\partial \beta_o^*}{\partial u_m}\right)\right]_{\bar{u}=\bar{u}_s} = 0 \tag{18}$$

‡ The motivation for the results presented here is given in Part 5. A partial proof of the second necessary and sufficient condition for the ordinary calculus is given in Appendix B

To establish a second necessary condition we construct the orthonormal (rotation) matrix L such that

$$[L]\left[\frac{\partial\bar{\beta}}{\partial\bar{\mu}}\right][K^*]^{-1}\left[\frac{\partial\bar{\beta}}{\partial\bar{\mu}}\right]'[L]' = [\text{diagonal}] \triangleq \begin{bmatrix}\rho_0 & & & 0\\ & \rho_1 & & \\ & & \ddots & \\ 0 & & & \rho_r\end{bmatrix} \quad (19)$$

Thus

$$(\bar{\eta}_i^*)'(K^*)^{-1}(\bar{\eta}_j^*) = \begin{cases}\rho_i & \text{if } i = j\\ 0 & \text{if } i \neq j\end{cases} \quad \cancel{(19)}$$

where
$$\left\{L\left[\frac{\partial\bar{\beta}}{\partial\bar{\mu}}\right]\right\}' \triangleq \left[\frac{\partial\bar{\beta}^*}{\partial\bar{\mu}}\right]' = [\bar{\eta}_0^* \vdots \bar{\eta}_1^* \vdots \cdots \vdots \bar{\eta}_r^*] \quad (20)$$

$$K^* = \left[\frac{\partial^2\beta_0^*}{\partial\mu_i\partial\mu_j}\right] \quad \cancel{(20)}\ (21)$$

and we have assumed [redacted] (that K* is non-singular) The first row of L is proportional to $\bar{\nu}$, and $\rho_0 = 0$ because of equation (18).

(close up) If we assume that all $\rho_i \neq 0$, for $i = 1, \ldots r$, the second necessary condition becomes

$$R = K^* - \sum_{i=1}^{r}\left(\frac{1}{\rho_i}\right)(\bar{\eta}_i^*)(\bar{\eta}_i^*)' \geq 0 \quad (22)$$

That is, R must be positive semi-definite. With the assumption that K^* is non-singular, [redacted] equation (22) also yields a sufficient condition. (If K^* is singular an equation analogous to (22) can be obtained.) In Appendix B it is shown that These conclusions [redacted] are equivalent to those presented in section 76 of reference 4. [redacted] This will be discussed further in a paper to be published.

The necessary and sufficient conditions for the continuous case are analogous. (reference 5) Given the r+1 functionals $\beta_i(\bar{x}\{t_1, t_2; u(t)\})$, where β_o is to be minimized, subject to $\beta_i = 0$, for i=1,...r, and β_k are linear functions of the $x_j(t_2)$, it can be shown that the ~~subject~~ first necessary condition is

$$\eta_o^*(t) = \sum_{i=0}^{r} \nu_i \eta_i^*(t) = 0 \qquad (23)$$

Equation (23) is equivalent to the Euler-Lagrange equation in the classical calculus of variations. A second necessary condition is

$$\xi_o^*(t) = \sum_{i=0}^{r} \nu_i \xi_i(t) \geq 0 \qquad (24)$$

which is the classical Legendre condition. It can be seen from equations (14) and (15) that $\eta_o^* = \left(\frac{\partial h}{\partial u}\right)$ and $\xi_o^* = \left(\frac{\partial^2 h}{\partial u^2}\right)$. Let us suppose that the inequality sign holds strictly in equation (~~19~~ 24) and, without further loss of generality, that $\xi_o^*(t) = 1$. This is accomplished by normalizing the impulse response functions by the factor $[\xi_o^*(t)]^{-\frac{1}{2}}$, which can always be done if $\xi_o^*(t) > 0$. Let $\{\omega_i\}_{i=1,\ldots\infty}$ be the eigenvalues of the real symmetric kernel

$$K^*(t,\tau) = \sum_{i=0}^{r} \nu_i \left[\xi_o^*(t)\,\xi_o^*(\tau)\right]^{-1/2} K_i(t,\tau) \qquad (25)$$

and let $\{\varphi_i(t)\}_{i=1,\ldots\infty}$ be the corresponding eigenfunctions, which we assume to be a complete in the sense of reference 6†. Again we construct an orthonormal transformation of the β_i to yield the analogue of equation (19), that is,

† The $\{\varphi_i(t)\}_{i=1\ldots\infty}$ may not be complete, as is shown on page 242 of reference 6. This situation corresponds to the matrix K* of equation (21) being singular. We deal with this case by arbitrarily adjoining additional orthonormal functions to complete the set. All of these new functions are orthogonal to the kernel $K^*(t, \tau)$, and hence have eigenvalues equal to zero.

$$\int_{t_1}^{t_2} \left[\sum_{i=1}^{\infty} \frac{d_{ij}\,\varphi_i(t)}{\omega_i + 1} \right] \left[\sum_{i=1}^{\infty} d_{ik}\,\varphi_i(t) \right] dt = \begin{cases} \rho_j & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases} \tag{26}$$

$$j, k = 0, 1, \cdots r$$

where

$$\rho_j = \sum_{i=1}^{\infty} \left(\frac{d_{ij}^2}{1 + \omega_i} \right) \qquad j = 0, 1, \cdots r \tag{27}$$

and $\{d_{ij}\}_{i=1,\ldots\infty}$ are the Fourier coefficients of the impulse response function $\eta_j{}^*(t)$, that is

$$\eta_j^*(t) \cong \sum_{i=1}^{\infty} d_{ij}\,\varphi_i(t) \tag{28}$$

Let us heuristically <u>define</u> the delta function kernel to be $\Delta(t, \tau) = \sum_{i=1}^{\infty} \varphi_i(t)\varphi_i(\tau)$, which has the same eigenfunctions as $K^*(t,\tau)$ but all eigenvalues equal to unity.† The analogue of equation (22) then becomes

$$R(t,\tau) = \Delta(t,\tau) + K^*(t,\tau) - \sum_{i=1}^{\infty} \left(\frac{1}{\rho_i} \right) \eta_i^*(t)\,\eta_i^*(\tau) \geq 0 \tag{29}$$

Thus R (t,τ) can have no eigenvalues less than zero, which is equivalent to saying that the kernel composed of the last two terms of equation (29) can have no eigenvalues less than -1. If the eigenfunctions of $K^*(t, \tau)$ do indeed form a complete set, (which implies that the inequality sign holds strictly in equation (29)), we have a sufficient condition.

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†A rigorous construction of this delta function will not be presented here.

To apply these results to our example problem we suppose that arc length (time of flight) is to be minimized at the fixed final longitude t_2, subject to $x_2(t_2) = 0$. Thus $\beta_o = x_1$, $\beta_2 = x_2$ and the standard trajectory is given by $x_{2s}(t) = 0$. In the appendix it is shown that $\nu = 0$, $\xi_o{}^*(t) = 1$, $\eta^*(t) = 1$, and the eigenvalues and eigenfunctions of $K^*(t, \tau)$ are computed. It is shown that the trajectory yields a minimum value of arc length if and only if $(t_2 - t_1) < \pi$, which is a well known result. If $(t_2 - t_1) > \pi$ the trajectory is neither maximizing nor minimizing.

5. A Geometrical Interpretation of Optimality

Let us seek a geometrical interpretation of the optimality conditions by defining an (r+1) dimensional Euclidean space with Cartesian coordinates given by $\delta\bar{\beta}\left\{t_1, t_2; u(t)\right\} = [\delta\beta_o, \delta\beta_1, \ldots \delta\beta_r]$. Thus any point in the space is a mapping of the function $\delta u(t)$ to the vector $\delta\bar{\beta}$, and the origin $\delta\bar{\beta} = 0$ corresponds to the optimal standard trajectory. Obviously all points in this space cannot be reached by varying the control, for then it would be possible to achieve $\delta\beta_o < 0$ and $\delta\beta_i = 0$ for $i = 1, \ldots r$, which contradicts the assumption that the standard trajectory was minimizing. Indeed, it can be shown (reference 7) that the envelope of reachable points is given, to second order, by the parabaloid (see figure 5)

$$\delta\beta_o^* - \sum_{i=1}^{r}\left(\frac{1}{\rho_i}\right)\left(\delta\beta_i^*\right)^2 = 0 \tag{30}$$

The orthonormal transformation L referred to in Part 4 has the effect of rotating the coordinate axes of the space to coincide with the principal axes of the reachable envelope, and the ρ_i are the radii of curvature at the origin. From equations (7) and (29) and the definition of the L matrix it follows that (to second order),

$$\int_{t_1}^{t_2}\int_{t_1}^{t_2} R(t,\tau)\,\delta u(t)\,\delta u(\tau)\,dt\,d\tau = \delta\beta_o^* - \sum_{i=0}^{r}\left(\frac{1}{\rho_i}\right)\left(\delta\beta_i^*\right)^2 \geq 0 \tag{31}$$

Thus the positive semi-definiteness of $R(t, \tau)$ implies that the reachable points always lie <u>above</u> the reachable envelope, which motivates the necessary and sufficient conditions described above. Various forms of reachable envelopes are shown in figure 6.

We have thus far assumed that $\eta_o^*(t) = 0$ but that none of the other impulse response functions are identically zero. If $\eta_i^*(t) = 0$ for $i = 1,...q$ the trajectory is said to be abnormal, of order q (reference 4, page 210)[‡], a troublesome case where the analysis discussed above does not directly apply. Abnormality is related to the concept of first order controllability, where a trajectory can be said to be first order uncontrollable of order p if there are p influence functions equal to zero. (references 7 and 8). The motivation for this definition follows from the observation that a control variation $\delta u(t)$ has no first order effect on $\delta\beta_i^*$ if $\eta_i^*(t) = 0$. Thus any optimal trajectory is first order uncontrollable of at least order one, and is first order uncontrollable of order q+1 if it is abnormal of order q.

The controllability definition can be extended by saying that a point $\delta\bar{\beta}^*$ is second order controllable if $-\delta\bar{\beta}^*$ lies within the reachable envelope. If this condition applies it is possible to find a control $\delta\tilde{u}(t)$ which results in $\Delta\bar{\beta}^* \triangleq \delta\bar{\beta}^*\{t_1, t_2; \delta\hat{u}(t)\} = -\delta\bar{\beta}^*$

‡ The definition of abnormality presented here is slightly different ~~than~~ from that presented in reference 4, where $\beta_o^* = \beta_1$ is also considered to be abnormal. This case corresponds to rotating the axes of the boundary function space (by the transformation L) through an angle of precisely $\pi/2$.

and hence $\Delta\bar{\beta}^* + \delta\bar{\beta}^* = 0$. For example, the control variation

$$\delta \hat{u}(t) = \epsilon_0 + \sum_{i=1}^{r} \epsilon_i \psi_i(t) \tag{32}$$

can realize any desired point within the reachable envelope if

$$\psi_i(t) = \sum_{j=1}^{\infty} \left(\frac{d_{ij}\, \varphi_j(t)}{1 + \omega_j} \right) \qquad i = 1, \cdots r$$

The $\psi_i(t)$ have the property that

$$\int_{t_1}^{t_2} R(t, \tau)\, \psi_i(t)\, dt = 0$$

that is, they form a set of r annihilator functions of the kernel $R(t, \tau)$. Thus equation (31) becomes

$$\delta\beta_0^* - \sum_{i=1}^{\infty} \left(\frac{1}{\rho_i}\right) \left(\delta\beta_i^*\right)^2 = \epsilon_0^2 \sum_{i=1}^{\infty} \omega_i \left[\int_{t_1}^{t_2} \varphi_i(t)\, dt\right]^2 \tag{33}$$

where

$$\delta\beta_i^* = \epsilon_0 \left[\int_{t_1}^{t_2} \eta_i^*(t)\, dt\right] + \sum_{i=1}^{r} \epsilon_i \rho_i \tag{34}$$

If the desired $\delta\bar{\beta}^*$ lies within the reachable envelope we know from equation (31) that the right hand side of equation (33) is positive.

Since we have assumed $\rho_i \neq 0$, it is thus always possible to find $\{\epsilon_i\}_{i=0,\ldots r}$ from equations (33) and (34) which will achieve any desired $\{\delta\beta_i{}^*\}_{i=0,1,\ldots r}$ within the reachable envelope. This analysis has important application to the guidance problem, where it is necessary to consider the end conditions which can be attained for any given initial condition disturbance, and to construct a control variation which will attain the desired result (reference 9)

We have thus far assumed that none of the radii of curvature of the reachable envelope are zero ($\rho_i \neq 0$ for all $i = 1, \ldots r$), which, in the terminology of classical calculus of variations, is equivalent to saying that the initial point t_1 and the final point t_2 are not "conjugate" to one another (reference 4). If one or more $\rho_i = 0$ the reachable envelope cannot be constructed, which implies that it is not possible to generate a family of minimizing trajectories in a small neighborhood of the standard trajectory which achieve slightly different end conditions. This notion is directly related to the classical concept of a "field" of extremals (reference 4). Let us imagine a family of standard trajectories, all extremals in the sense that $\eta_o{}^*(t) = 0$ along any trajectory (that is, the Euler-Lagrange equations are satisfied along every path). The initial state vectors $\bar{x}_{1s}$ of this family lie on a given n-dimensional manifold, the final state vectors $\bar{x}_{2s}$ lie on some other given n-dimensional manifold, and the initial and final times range over the values $0 \leq t_1 < t_2 \leq T$.

For each trajectory we proceed with the analysis described above, and generate a family of symmetric kernels which are functions not only of (t, τ) but also of $(t_1, t_2, \bar{x}_{1s}, \bar{x}_{2s})$, that is, $R = R(t, \tau, t_1, t_2, \bar{x}_{1s}, \bar{x}_{2s})$. If the corresponding radii of curvature $\rho_i(t_1, t_2, \bar{x}_{1s}, \bar{x}_{2s}) \neq 0$ for $i = 1, \ldots r$, it can be shown that the family of extremal trajectories is a field of extremals in the classical sense. ~~It is shown in the appendix that~~ Such a field can be constructed for the example problem discussed in the appendix if and only if the arc length is less than π.

(insert figs #5 and 6 here)

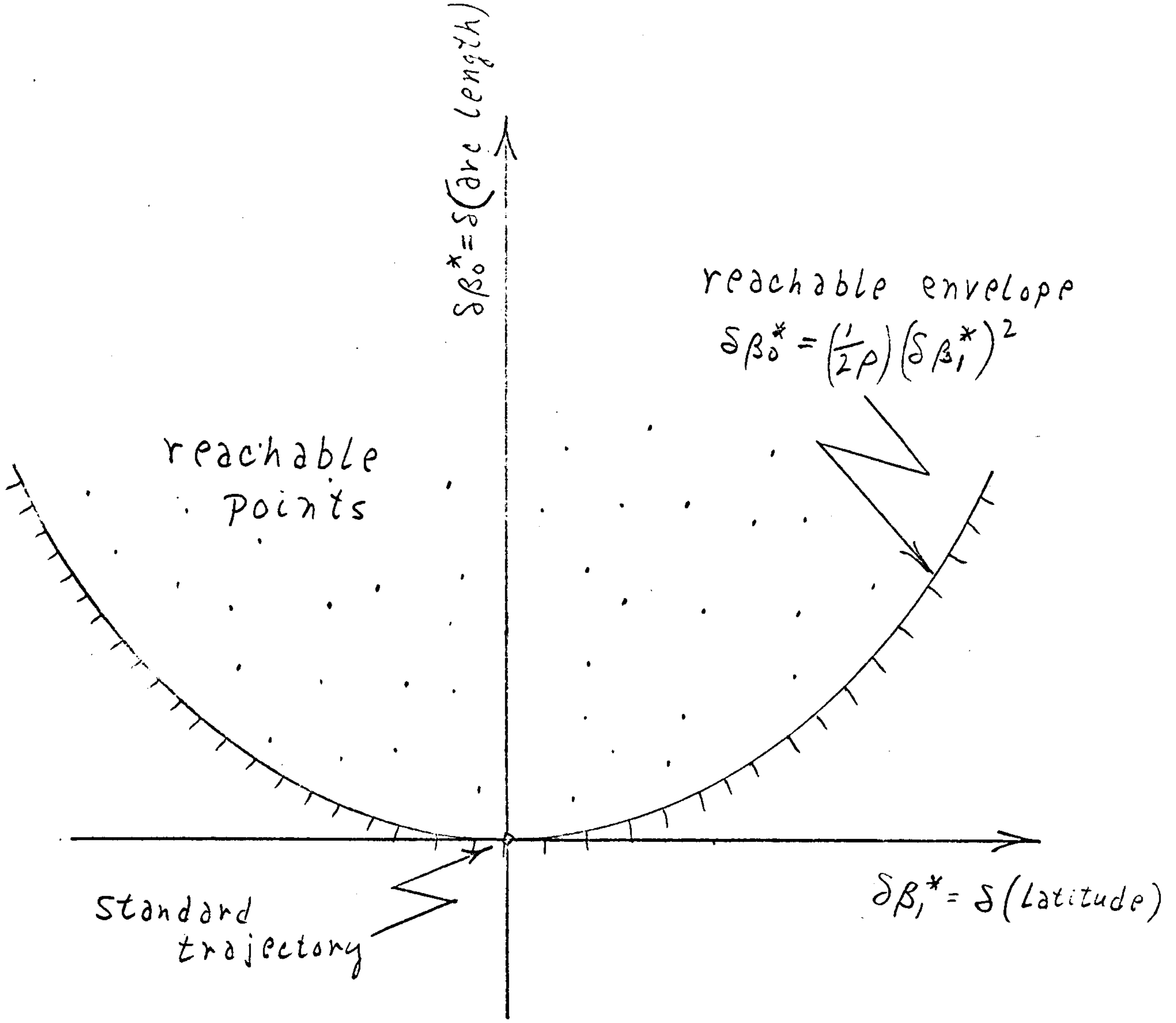
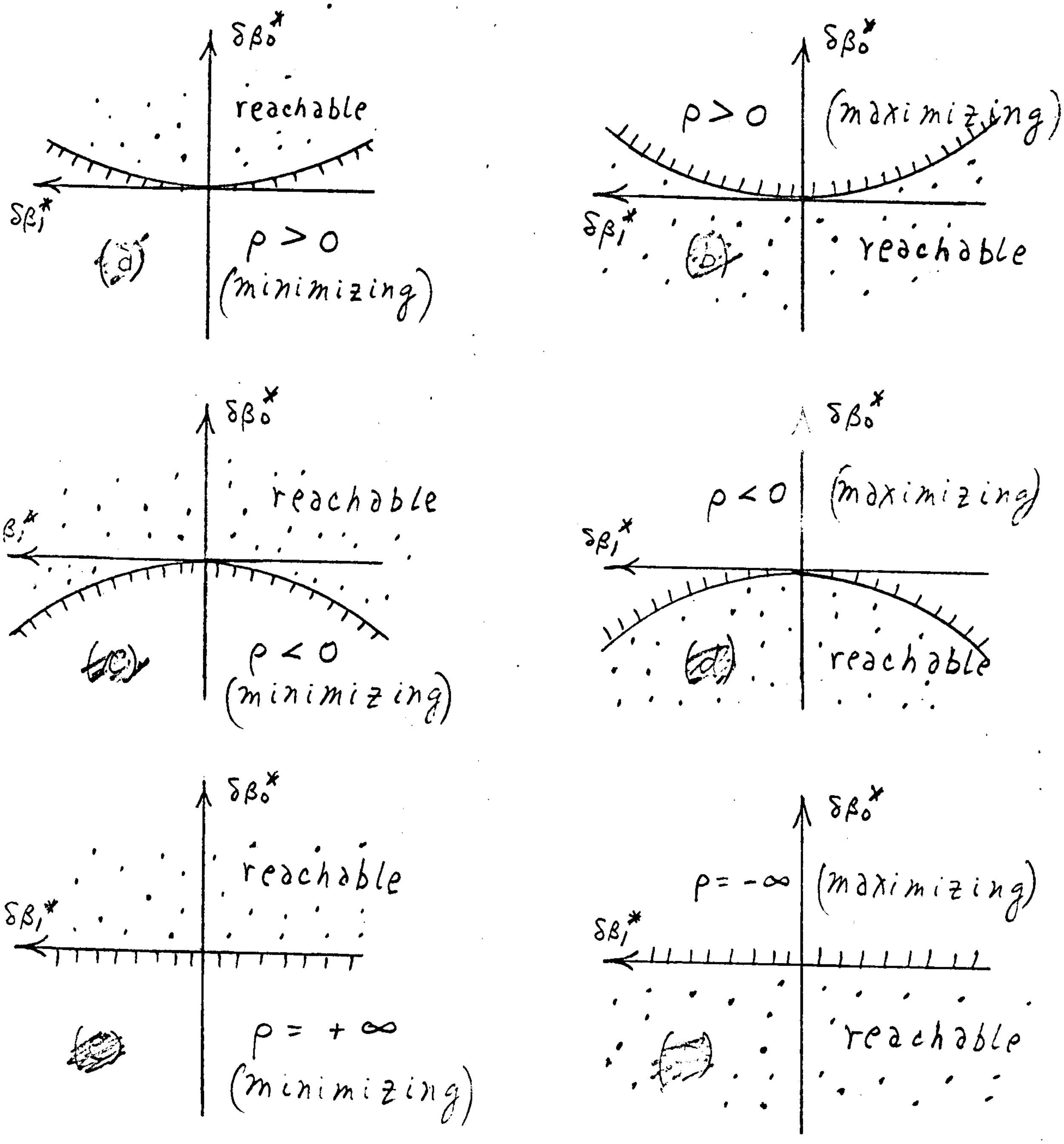


Fig 5: A GEOMETRICAL INTERPRETATION OF OPTIMALITY



F I G. 6: Various Types of Reachable Envelopes

6. Optimal Control in the Presence of Random Disturbances

In the preceeding discussion of optimal control it was assumed that the accelerations acting on the spacecraft are always known, while in fact this is usually not the case. Suppose, for example, that the equation of motion (1) takes the form

$$\frac{d}{dt}\bar{x} = \bar{f}(\bar{x}, u, t) + \bar{g}(t) \tag{35}$$

where $\bar{g}(t)$ is a random vector function of time. Typical causes of such random forcing functions would be unknown solar winds, and/or non-standard performance of the spacecraft propulsion system. To deal with this case we must seek a meaningful way to define an optimal control in the presence of random disturbances.

Let us assume that the statistical description of $\bar{g}(t)$ is known, for example, suppose that $\bar{g}(t) = B(t)\,\bar{\alpha}$, where the α_i are zero mean Gaussian random variables with known variances and correlations. The sample space associated with $\bar{\alpha}$ is taken to be the ensemble composed of an infinite number of space missions with the same standard trajectory and mission objectives, but with values of α_i randomly selected from the given multivariate Gaussian distribution. Any space mission would not actually be repeated many times, of course, but this ensemble makes sense if we imagine a large number of numerical <u>simulations</u> of the mission on a digital computer, with Monte Carlo selection of the α_i. The guidance system performance index will be taken as the expected value (statistical expectation) of the quantity to be minimized in the deterministic case, evaluated by the computer simulation or some equivalent method. This approach is intuitively satisfying, for if the system performs adequately for a large percentage of the cases numerically simulated, it is reasonable to say that it has been properly designed for a single mission.

Let us examine the task of constructing the optimal control function by considering a simple problem in the ordinary calculus. Suppose we are to minimize (in some sense) the single function $\beta(u, \alpha)$, where u is the control variable and α is a zero-mean Gaussian random variable with variance σ^2. We expand in Taylor's series about the point $(u = u_s, \alpha = 0)$ to obtain

$$\begin{aligned}\delta\beta &= \beta(u, \alpha) - \beta(u_s, 0)\\ &= \left(\frac{\partial\beta}{\partial u}\right)\delta u + \left(\frac{\partial\beta}{\partial\alpha}\right)\alpha + \left(\frac{\partial^2\beta}{\partial u\,\partial\alpha}\right)\delta u\,\alpha\\ &\quad + \frac{1}{2}\left(\frac{\partial^2\beta}{\partial u^2}\right)\delta u^2 + \frac{1}{2}\left(\frac{\partial^2\beta}{\partial\alpha^2}\right)\alpha^2 + \text{higher order terms} \qquad (36)\end{aligned}$$

where the partial derivatives are evaluated at $(u = u_s, \alpha = 0)$. We find the expected value of $\delta\beta$) to be

$$
\begin{aligned}
E[\delta\beta] &= \int_{-\infty}^{\infty} \delta\beta(u,\alpha)\, p(\alpha)\, d\alpha \\
&= \left(\frac{\partial\beta}{\partial u}\right)\delta u + \frac{1}{2}\left[\left(\frac{\partial^2\beta}{\partial u^2}\right)\delta u^2 + \left(\frac{\partial^2\beta}{\partial\alpha^2}\right)\sigma^2\right] \\
&\quad + \frac{1}{3!}\left[\left(\frac{\partial^3\beta}{\partial u^2}\right)\delta u^3 + 3\left(\frac{\partial^3\beta}{\partial u\,\partial\alpha^2}\right)\sigma^2\,\delta u\right] \\
&\quad + \frac{1}{4!}\left[\left(\frac{\partial^4\beta}{\partial u^4}\right)\delta u^4 + 6\left(\frac{\partial^4\beta}{\partial u^2\,\partial\alpha^2}\right)\sigma^2\,\delta u^2 + 3\left(\frac{\partial^4\beta}{\partial\alpha^4}\right)\sigma^4\right] \qquad (37) \\
&\quad + \text{higher order terms}
\end{aligned}
$$

where $p(\alpha)$ is the (Gaussian) probability density function for α, and $E[\ldots]$ indicates the statistical expectation of the quantity in brackets. To obtain equation (37) we have used $E[\delta\alpha] = E[\delta\alpha^3] = \ldots\ldots = E[\delta\alpha^{2n+1}] = 0$, $E[\delta\alpha^2] = \sigma^2$, and $E[\delta\alpha^4] = 3\sigma^4$, which are properties of the Gaussian distribution. Just as in the deterministic case, we seek to minimize the expected value of $\delta\beta$ by searching for a u_s which sets the coefficient of δu equal to zero equation (37). Thus,

$$
\left[\left(\frac{\partial\beta}{\partial u}\right) + \left(\frac{\partial^3\beta}{\partial u\,\partial\alpha}\right)\left(\frac{\sigma^2}{2}\right) + \ldots\ldots\right] = 0 \qquad (38)
$$

where all the $(2n + 1)$st derivatives of $\beta(u, \alpha)$ contribute to the "first variation" with respect to u, that is, terms of the form $\left(\frac{\partial^{2n+1}\beta}{\partial u\,\partial\alpha^{2n}}\right) E[\alpha^{2n}]$ appear in equation (38).

To obtain an approximation to equation (38) we might only insist that $\left[\frac{\partial\beta}{\partial\mu}(\mu_s,0)\right]=0$, hoping that the higher derivative terms are negligble (an assumption which is not always justified). Because of the difficulty of dealing with all the variations with respect to all of the random elements of a system of the form (35), this simplified approach is often employed when constructing the optimal control in the presence of random disturbances. Thus we replace the random function $\bar{g}(t)$ in equation (35) with its (time varying) expected value $E[\bar{g}(t)]$, and proceed as in the deterministic case. For those situations where this simplication is not justified it often becomes quite difficult to find a solution by classical methods, and the dynamic programming technique becomes an attractive analytical tool (reference 10, chapter 10). An example of a solution to such a problem based upon the dynamic programming point of view is given in reference 11.

7. Sequential Estimation of the State Vector

In order to obtain the initial conditions for the optimal control equations we come to the navigation problem, where the state vector describing the trajectory must be estimated from noise contaminated tracking data. To simplify the discussion we shall assume that the accelerations acting upon the spacecraft are known, so that the six dimensional state vector $\bar{x}(t_i)$ composed of the position and velocity components at any time t_i

is sufficient to determine the trajectory for all future time $t \geq t_i$.

Assume that we are given a sequence of q dimensional data vectors $\{\bar{\varphi}(t_1), \bar{\varphi}(t_2) \dots \varphi(t_i) \dots\}$ related to the time varying state vector, but which are contaminated with additive noise (measurement error). Comparing the given $\bar{\varphi}(t_i)$ with the values that would be observed if the trajectory were standard and the data noise were zero, we have, to first order

$$\delta\bar{\varphi}(t_i) = \bar{\varphi}(t_i) - \bar{\varphi}_s(t_i) = \left[\frac{\partial\bar{\varphi}(t_i)}{\partial\bar{x}(t_i)}\right]\delta\bar{x}(t_i) + \bar{n}(t_i) \triangleq A_i\,\delta\bar{x}_i + \bar{n}_i \tag{39}$$

where $\bar{n}(t_i)$ is a q dimensional noise vector and A_i is the partial derivative matrix given in equation (39). (We shall henceforth use subscripts to denote times t_i) Again we consider the ensemble of all varied trajectories defined by the Monte Carlo computer simulation discussed in Part 6, and we assume that over this ensemble we know the statistical description of the random variables $\delta\bar{x}_i$ and $\bar{n}_i$. Suppose that at some time t_i we have a minimum variance estimate of the first variation of the state vector at t_i, denoted by $\delta\bar{x}_i^*$, which is based upon all data up to and including time t_i. The minimum variance estimation technique (reference 12) has the property that the variance of the error in the estimate is a minimum when compared to the error variance obtained with any other linear, unbiased estimate. We define the error in the estimate to be $\bar{\epsilon}_i = [\delta\bar{x}_i^* - \delta\bar{x}_i]$, and the error covariance matrix to be

$$\Lambda_i^* = E[\bar{\epsilon}_i\ \bar{\epsilon}_i{}'] \tag{40}$$

Seeking a sequential (iterative) technique for treating the data, we <u>advance</u> the estimate and covariance matrix from t_i to t_{i+1} according to[‡]

$$\delta \hat{\bar{x}}_{i+1} = U_{(i+1,\ i)}\ \delta \bar{x}^*_i \tag{41}$$

$$\hat{\Lambda}_{i+1} = \left[U_{(i+1,\ i)}\right]\left[\Lambda^*_i\right]\left[U'_{(i+1,\ i)}\right] \tag{42}$$

where the ($\wedge$) denotes the advanced quantity. We <u>update</u> the advanced estimate by incorporating the new data point at t_{i+1} according to

$$\delta \bar{x}^*_{i+1} = W_{i+1}\left[\delta \bar{\varphi}_{i+1} - A_{i+1}\,\delta \hat{x}_{i+1}\right] + \delta \hat{\bar{x}}_{i+1} \tag{43}$$

where

$$W_{i+1} = \hat{\Lambda}_{i+1} A'_{i+1}\left[A_{i+1}\hat{\Lambda}_{i+1}A'_{i+1} + \Gamma_{i+1}\right]^{-1} \tag{44}$$

$$\Gamma_{i+1} = E\left[\bar{n}_{i+1}\,\bar{n}'_{i+1}\right] \tag{45}$$

The updated covariance matrix becomes

$$\Lambda^*_{i+1} = \left[I - W_{i+1} A_{i+1}\right]\hat{\Lambda}_{i+1} \tag{46}$$

---

‡ See Part 3 for definition of $U_{(i+1,\ i)} = U(t_{i+1},\ t_i)$

In the same fashion we proceed to observation times t_{i+2}, t_{i+3}, ... and sequentially treat the data vector at each time in the manner described above. The process is started at the initial observation time t_1 by taking $\hat{x}_1 = 0$ and $\hat{\Lambda}_1$ = the a-priori uncertainty. This approach to the estimation problem was developed by Kalman in his analysis of linear dynamic systems excited by uncorrelated error sources (reference 13).

As a simple illustration of the method let us consider the problem of estimating the unknown constant δx when given the observations

$$\delta\varphi(t_i) = \delta x + n(t_i) \tag{47}$$

where $E[\delta x^2] = \sigma_x^2$, $E[n^2(t_i)] = \sigma_n^2$, and $E[\delta n_i \delta n_j] = 0$ for $i \neq j$. Thus equations (41) - (46) become

$$\delta\hat{x}_{i+1} = \delta x_i^* \tag{48}$$

$$(\hat{\sigma}_{i+1})^2 = (\sigma_i^*)^2 \tag{49}$$

$$\delta x_{i+1}^* = (\sigma_i^*)^2 \left[(\sigma_i^*)^2 + (\sigma_n)^2\right]^{-1} \left[\delta\varphi_{i+1} - \delta x_i^*\right] + \delta x_i^* \tag{50}$$

$$\begin{aligned}(\sigma_{i+1}^*)^2 &= (\sigma_i^*)^2 - (\sigma_i^*)^4 \left[(\sigma_i^*)^2 + (\sigma_n)^2\right]^{-1} \\ &= (\sigma_i^*)^2 (\sigma_n)^2 \left[(\sigma_i^*)^2 + (\sigma_n)^2\right]^{-1}\end{aligned} \tag{51}$$

where the initial values are $\delta\hat{x}_1 = 0$ and $(\hat{\sigma}_1)^2 = (\sigma_x)^2$. Note that equation (50) can also be written as

$$\delta x^*_{i+1} = \left[(\sigma^*_i)^{-2} + (\sigma_n)^{-2}\right]^{-1}\left[(\delta\varphi_{i+1})(\sigma_n)^{-2} + (\delta x^*_i)(\sigma^*_i)^{-2}\right] \quad (52)$$

and equation (51) can also be written as

$$(\sigma^*_{i+1})^2 = [(\sigma^*_i)^{-2} + (\sigma_n)^{-2}] \quad (53)$$

Equation (52) is the well-known formula for combining two uncorrelated estimates with different variances, and equation (53) is the well-known formula for the resulting variance of the combined estimate.

8. Continuous Estimation of the State Vector

As the time intervals $(t_{i+1} - t_i)$ become small it is reasonable to ask if a continuous form of the estimation equations can be obtained, that is, we seek to replace the difference equations for the estimate and covariance matrix with differential equations. An analysis of the continuous case by Kalman (Reference 13) is based upon the assumed existence of a differential equation describing the state vector to be estimated, where there is a continuous, "white noise" random forcing function. In the classical analysis of continuous Markoff processes this would be analogous to the Langevin equation. (reference 14) We shall take a different approach here, considering the sequential estimation equations appropriate to the most general type of stochastic process, where observations are taken at discrete times, and extend the result to the continuous case without constructing a dynamic model of the process (reference 15).

Let us imagine a stochastic process which is composed of an infinite number of time records of the n-dimensional random vector $\bar{y}(t)$, that is, we have the ensemble $\{\bar{y}^{\alpha}(t)\}$ for $\alpha = 1, \cdots \infty$. We assume that the a priori first and second moments of the process are known to be $E[\bar{y}(t)] = 0$, $E[\bar{y}(t_i)\bar{y}'(t_i)] = \Lambda(t_i) = \Lambda_i$, and $E[\bar{y}(t_j)\bar{y}'(t_i)] = C(t_j, t_i) = C_{ji}$, where the statistical expectations are taken over the the ensemble $\{\bar{y}^{\alpha}(t)\}$. On some one experiment, corresponding to a single (unknown) record $\bar{y}^k(t)$, we observe a sequence of data vectors $\{\delta\bar{\varphi}_1, \delta\bar{\varphi}_2, \cdots, \delta\bar{\varphi}_i, \cdots\}$ at the discrete times $\{t_1, t_2, \cdots t_i \cdots\}$. We assume that the $\delta\bar{\varphi}_i$ are linearly related to $\bar{y}^k(t)$, that is,

$$\delta\varphi_i = B(t_i)\,\bar{y}^k(t_i) \triangleq B_i\,\bar{y}_i^k \tag{54}$$

where the B_i are known matrices. In general, we cannot employ a sequential estimation technique to obtain the minimum variance estimate $\bar{y}^*(t)$ for this correlated stochastic process, because the information contained in the data up to and including time t_i cannot always be represented by the n by n covariance matrix Λ_i^*. Indeed, it is shown in Reference 12 that a necessary, and sufficient condition to be satisfied by process correlation matrices if the sequential estimation approach to be justified is

$$P_{ki} = P_{kj} P_{ji} \qquad \text{for all } t_i \le t_j \le t_k \tag{55}$$

where

$$P_{ji} = (C_{ji})(\Lambda_i^{-1}) \tag{56}$$

The P_{ji} is called the "normalized correlation matrix", and, if equation (55) applies, the process is said to be "sequentially correlated". Equations (41) - (46) then generalize to

$$\hat{\bar{y}}_{i+1} = (P_{i+1,i})(\bar{y}_i^*) \tag{55}$$

$$\hat{\Lambda}_{i+1} = \Lambda_{i+1} - (P_{i+1,i})(\Lambda_i - \Lambda_i^*)(P'_{i+1,i}) \tag{56}$$

$$\bar{y}^*_{i+1} = W_{i+1}(\delta\varphi_{i+1} - B_{i+1}\hat{\bar{y}}_{i+1}) + \hat{\bar{y}}_{i+1} \tag{57}$$

$$W_{i+1} = (\hat{\Lambda}_{i+1} B_{i+1})(B_{i+1}\hat{\Lambda}_{i+1} B'_{i+1})^{-1} \tag{58}$$

$$\Lambda^*_{i+1} = (I - W_{i+1} B_{i+1})(\hat{\Lambda}_{i+1}) \tag{59}$$

To show the correspondence to the application discussed in Part 7, we let $\bar{y}_i' = [\delta\bar{x}_i', \bar{n}_i]$, that is, we incorporate the noise vector as part of the state vector. By defining $B_i = [A_i \vdots I]$, where I is the q by q identity matrix, we have equation (54). Suppose the noise is exponentially correlated, that is,

$$E[\bar{n}(t_i)\,\bar{n}'(t_j)] = [\exp D(t_j - t_i)]\, E[\bar{n}_i \bar{n}_i'] \qquad t_j \ge t_i$$

where D is a constant matrix. Then from the definition of P_{ji} it is easy to show that

$$P_{ji} = \left[\begin{array}{c|c} U_{ji} & O \\ \hline O & \exp D(t_j - t_i) \end{array}\right] \qquad t_j \ge t_i \tag{60}$$

and equations (41) - (46) follow from (55) - (59).

A sequentially correlated process has the property that the minimum variance estimate of a future state depends only upon the present estimate and error covariance matrix, but not upon the past history of the process. Such a process can be thought of as a generalized, or "wide sense", Markoff process, which is similarly defined but in terms of the conditional probability of the future state. Thus the sequential correlation definition treats only the first two moments of the process, while the Markoff definition implicity deals with all the moments. If the components of $\bar{y}_i$ are Gaussian variables, so that the first and second moments completely specify the process, then the two definitions become equivalent.* This more general approach to the estimation problem seems to hold only academic interest for the discrete case, for, as pointed out above, whether we postulate a dynamic model or the sequential correlation condition, we come to the same result when applying the technique to the trajectory problem. When extending the sequential estimation technique to the continuous case, however, we come to different results from the two sets of assumptions.

The continuous equations are obtained in Reference (15) by expanding the quantities appearing in equations (55) - (59) in Taylor's series about an arbitary point t_i. Letting $t_{i+1} \rightarrow t_i$, we obtain

$$\frac{d}{dt}\bar{y}^* = W\bar{y}^* + Q\left(\frac{d\Phi}{dt}\right) \tag{61}$$

$$\frac{d}{dt}\Lambda^* = S - Q\left[\left(\frac{dB}{dt}\right)\Lambda^* + BS\right] \tag{62}$$

where the (time dependent) elements of equations (61) and (62) are

$$S = \left(\frac{d\Lambda}{dt}\right) + \left(\Lambda^*-\Lambda\right)\left(\frac{dP}{dt}\right)' + \left(\frac{dP}{dt}\right)\left(\Lambda^*-\Lambda\right) \tag{63}$$

---

* In Reference 16, it is shown that any stationary, Gaussian Markoff process is sequentially correlated.

$$\left(\frac{dP}{dt}\right) = \left[\frac{\partial P(t_j, t_i)}{\partial t_j}\right]_{t_i = t_j = t} \tag{64}$$

$$Q = \left[\Lambda^* \left(\frac{dB}{dt}\right)' + S B'\right] \left[B S B'\right]^{-1} \tag{65}$$

$$W = \left[I - QB\right]\left[\frac{dP}{dt}\right] - Q\left(\frac{dB}{dt}\right) \tag{66}$$

The initial conditions for equations (61) and (62) are

$$\bar{y}_0^* = \Lambda_0 B_0' \left[B_0 \Lambda_0 B_0'\right]^{-1} \delta \bar{\varphi}_0 \tag{67}$$

$$\Lambda_0^* = \left[I - \Lambda_0 B_0' \left(B_0 \Lambda_0 B_0'\right)^{-1} B_0\right] \left(\Lambda_0\right) \tag{68}$$

Equations (61) and (62) can be numerically integrated to obtain $\bar{y}^*(t)$ and $\Lambda^*(t)$.

As an illustration of the continuous estimation technique, let us treat the continuous version of the simple problem discussed in Part 7. Letting $\delta x = y_1$ and $n(t) = y_2(t)$, we have

$$\delta\varphi(t) = y_1 + y_2(t) \tag{69}$$

where y_1 is an unknown constant to be determined, with a priori variance σ_x^2, and $y_2(t)$ is noise, with autocorrelation function given by

$$E\left[y_2(t_j)\, y_2(t_i)\right] = \sigma_n^2 \exp\left[-\kappa(t_j - t_i)\right] \qquad t_j \geq t_i \tag{70}$$

Thus

$$B'(t) = [1, 1] \tag{71}$$

$$P_{ji} = \begin{bmatrix} 1 & 0 \\ 0 & \exp\left[-\kappa(t_j - t_i)\right] \end{bmatrix} \qquad t_j \geq t_i \tag{72}$$

$$\frac{dP}{dt} = \begin{bmatrix} 0 & 0 \\ 0 & -\kappa \end{bmatrix} \tag{73}$$

This problem is solved in closed form in Reference (15). It is shown that the differential equation for the estimate is

$$\frac{d}{dt}\begin{bmatrix} y_1^* \\ y_2^* \end{bmatrix} = \left(\frac{c\, y_2^*}{q}\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \left(\frac{\frac{d\phi}{dt}}{q}\right)\begin{bmatrix} q-1 \\ 1 \end{bmatrix} \tag{74}$$

where

$$q(t) = 2\left[\left(\frac{\sigma_n}{\sigma_x}\right)^2 + \frac{ct}{2} + 1\right] \tag{75}$$

Equation (74) can be integrated to yield

$$y_1^*(t) = \left(\frac{\delta\varphi(t)}{q(t)}\right) + \left(\frac{\delta\varphi(0)}{q(0)}\right)\left[2\left(\frac{\sigma_n}{\sigma_x}\right)^2 + 1\right] + \left(\frac{c}{q(t)}\right)\left[\int_0^t \delta\varphi(\tau)\, d\tau\right] \tag{76}$$

$$y_2^*(t) = \delta\varphi(t) - y_1^*(t) \tag{77}$$

where

$$E\left[(y_1^*(t) - y_1)^2\right] = (2\,\sigma_n^2)(q(t))^{-1} \tag{78}$$

It follows that as $ct \rightarrow \infty$, we have

$$y_1^*(t) \rightarrow \left(\frac{1}{t}\right)\left[\int_0^t \delta\varphi(\tau)\, d\tau\right] \tag{79}$$

$$E\left[(y_1^*(t) - y_1)^2\right] \rightarrow (2\,\sigma_n^2)(ct)^{-1} \tag{80}$$

It is interesting to compare equations (79) and (80) to the limiting values of the estimate and error variance obtained in the discrete case discussed in Part 7, where, for large numbers of measurements (n), we have

$$\delta x^* \longrightarrow \left(\frac{1}{n}\right)\sum_{i=1}^{n} \delta\varphi_i \tag{81}$$

$$E\left[(\delta x^* - \delta x)^2\right] \rightarrow \left(\frac{1}{n}\right)(\sigma_n^2) \tag{82}$$

If we decompose the interval $(0, t)$ into n increments $\Delta t = \left(\frac{t}{n}\right)$, and represent equation (79) by the sum

$$\left(\frac{1}{t}\right)\left[\int_0^t \delta\varphi(\tau)\,d\tau\right] \approx \left(\frac{\Delta t}{t}\right)\sum_{i=1}^{n} \delta\varphi_i = \left(\frac{1}{n}\right)\sum_{i=1}^{n} \delta\varphi_i \tag{83}$$

Equation (80) becomes

$$\left(\frac{2\,\sigma_n^2}{\lambda t}\right) = \left(\frac{1}{n}\right)\left(\frac{2\,\sigma_n^2}{\lambda \Delta t}\right) \tag{84}$$

Thus the discrete and continuous cases have the same limits if we interpret the equivalent uncorrelated "white noise" variance in the discrete case to be $\left(\sigma_n^2\right)\left(\frac{\lambda \Delta t}{2}\right)^{-1}$. Note that the equivalent white noise variance goes to infinity as Δt goes to zero, which is to be expected.

Conclusion

We have presented here a theoretical discussion of the guidance and navigation problem, treated from the point of view of optimal control and estimation theory. The approach has been somewhat simplified and heuristic, based upon analytical results recently obtained (references 5, 7, 12 and 15). A more rigorous and complete discussion of some of the ideas introduced here will be published in the near future.

Appendix: Optimal Control on the Unit Sphere

In this appendix the analysis of the example problem ~~discussed~~ introduced in Part 2 will be developed, following the discussions in Parts 3 and 4. The equations of motion are given by

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} (u^2 + \cos^2 x_2)^{\frac{1}{2}} \\ u \end{bmatrix} = \bar{f}(\bar{x}, u) \qquad \text{(A.1)}$$

where $u(t)$ is the control function. Assuming $u_s(t) = 0$, the standard trajectory is given by $x_{2s}(t) = 0$, and:

$$F(t) = \left[\frac{\partial \bar{f}}{\partial \bar{x}}\right] = \begin{bmatrix} 0 & \left(\dfrac{-\cos x_{2s} \sin x_{2s}}{\sqrt{u_s^2 + \cos^2 x_{2s}}}\right) \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{(A.2)}$$

$$G(t)' = \left(\frac{\partial \bar{f}}{\partial u}\right)' = \left[\frac{u_s}{\sqrt{u_s^2 + \cos^2 x_{2s}}}, 1\right] = [0, 1] \qquad \text{(A.3)}$$

$$H(t)' = \left(\frac{\partial^2 f}{\partial u^2}\right) = [1, 0] \qquad \text{(A.4)}$$

$$J_1(t) = \begin{bmatrix} \left(\dfrac{\partial^2 f_1}{\partial x_1^2}\right) & \left(\dfrac{\partial^2 f_1}{\partial x_1 \partial x_2}\right) \\ \\ \left(\dfrac{\partial^2 f_1}{\partial x_1 \partial x_2}\right) & \left(\dfrac{\partial^2 f_1}{\partial x_2^2}\right) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \\ 0 & -1 \end{bmatrix} \tag{A.5}$$

$$J_2(t) = 0 \tag{A.6}$$

$$M(t) = 0 \tag{A.7}$$

It follows from equation (A.2) that

$$U(t, \tau) = \begin{cases} \text{the identity for } t \geq \tau \\ 0 \text{ for } t < \tau \end{cases} \tag{A.8}$$

Given $\beta_i = \bar{a}_i' \, \bar{x}(t_2)$ for $i = 0, 1, \ldots r$, we have

$$\bar{\lambda}_i'(t) = \bar{a}_i' \, U(t_2, t) = (a_{1i}, a_{2i}) \tag{A.9}$$

$$\eta_i(t) = \bar{\lambda}_i'(t) \, G(t) = a_{2i} \, , \tag{A.10}$$

$$\xi_i(t) = \bar{a}' \, H(t) = a_{1i} \tag{A.11}$$

Thus we have

$$\delta\beta_i(T) = a_{1i}\delta x_1(t_1) + a_{2i}\delta x_2(t_1) + a_{2i}\int_{t_1}^{t_2} \delta u(t)\ dt$$

$$+\left(\frac{a_{1i}}{2}\right)\left[\int_{t_1}^{t_2} \delta u^2(t)\ dt + \int_{t_1}^{t_2}\int_{t_1}^{t_2} K^*(t, \tau)\delta u(t)\ \delta u(\tau)\ dt d\tau\right]$$

$$+ \text{ higher order terms } i = 0, 1, \ldots r \tag{A.12}$$

where

$$K^*(t, \tau) = \begin{cases} (t - T) \text{ for } t \geq \tau \\ (\tau - T) \text{ for } \tau > t \end{cases} \tag{A.13}$$

Following the discussion of Part 4, we seek to minimize $\beta_o = x_1(t_2)$, subject to $\beta_1 = x_2(t_2) = 0$. Thus $\bar{a}_o' = (1, 0)$, $\bar{a}_1' = (0, 1)$, and equations (12) and (13) are obtained. It follows that the Lagrange multiplier is $\nu = 0$, and that $u_s(t) = 0$ is the standard control. We find the eigenfunctions and eigenvalues of $K^*(t, \tau)$ by twice differentiating the equation

$$\omega_i\varphi_i(t) = \int_{t_1}^{t_2} K^*(t, \tau)\ \varphi_i(\tau)\ d\tau \tag{A.14}$$

Thus

$$\omega_i = -\pi^2\left[\frac{2i - 1}{2(t_2 - t_1)}\right]^2 \qquad i = 1, \ldots \infty \tag{A.15}$$

$$\varphi_i(t) = \left(\frac{2}{t_2-t_1}\right)^{\frac{1}{2}} \cos(-\omega_i)^{\frac{1}{2}}(t-t_1) \tag{A.16}$$

$$i = 1, \ldots \infty$$

It can be shown that the $\left\{\varphi_i(t)\right\}_{i=1,\ldots\infty}$ do indeed form a complete set over the interval (t_1, t_2). The radius of curvature ρ is given by[‡]

$$\rho = (t_2 - t_1)\left\{1 - \left(\frac{8}{\pi^2}\right)\sum_{i=1}^{\infty}[2i-1]^{-2}\,[1+\omega_i]^{-1}\right\}$$

$$= \tan(t_2 - t_1) \tag{A.17}$$

The eigenvalues $\widetilde{\omega}_i$ of the kernel $R(t, \tau)$ are solutions of the equation

$$(1 - \widetilde{\omega}_i)^{\frac{1}{2}} \tan(1 - \widetilde{\omega}_i)^{\frac{1}{2}}\,(t_2 - t_1) = \tan(t_2 - t_1) \tag{A.18}$$

Thus we conclude that all $\widetilde{\omega}_i \geq 0$ if and only if $(t_2 - t_1) \leq \pi$.

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‡ See Titchmarsh, "The Theory of Functions," to evaluate the series of (A.17).

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APPENDIX B: PROOF OF THE SECOND NECESSARY CONDITION FOR OPTIMALITY IN THE ORDINARY CALCULUS ‡

In theorem 76.2 of reference 4 it is shown that the second necessary condition for optimality in the ordinary calculus is

$$\delta\bar{u}'[K^*]\delta\bar{u} \geq 0 \tag{B.1}$$

for all $\delta\bar{u}$ satisfying

$$N'\delta\bar{u} = 0 \tag{B.2}$$

where

$$N = [\bar{n}_1^* \,\vdots\, \bar{n}_2^* \,\vdots\, \cdots \,\vdots\, \bar{n}_r^*] \tag{B.3}$$

and the notation of Part 4 has been employed in equations (B.1)–(B.3). Suppose we let $A = \begin{bmatrix} A_1 \\ \hline A_2 \end{bmatrix}$ be the (m by m) orthonormal matrix which diagonalizes $[N N']$. From equation B.2 we have

$$0 = [ANN'A']A\delta\bar{u} = \begin{bmatrix} (A_1 N A_1') & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta\bar{u}_1^* \\ \delta\bar{u}_2^* \end{bmatrix} \tag{B.4}$$

where A_1 is the upper (r by m) submatrix

‡ This proof was suggested by Dr. W. G. Melbourne of the Jet Propulsion Laboratory

of A, A_2 is the lower ($m-r$ by m) submatrix of A, and

$$\delta \bar{u}_1^* = A_1 \, \delta \bar{u} \qquad (B.5)$$

$$\delta \bar{u}_2^* = A_2 \, \delta \bar{u} \qquad (B.6)$$

The $[NN']$ matrix has rank r, and we have arbitrarily rearranged terms to obtain equation (B.4) in the form given. We see that equation (B.4) requires that $\delta \bar{u}_1^* = 0$, and hence equation (B.1) becomes

$$(\delta \bar{u}' A')[A K^* A'] A \, \delta \bar{u} = (\delta \bar{u}_2^*)'[A_2 K^* A_2'](\delta \bar{u}_2^*) \geqq 0 \quad \text{for all } \delta \bar{u}_2^* \qquad (B.5)$$

Thus the second necessary condition of reference 9 is $[A_2 K^* A_2'] \geqq 0$

We consider now the matrix R of equation (22), and define the vectors

$$\bar{\Psi}_i = [K^*]^{-1} \bar{\eta}_i^* \qquad i = 1, \ldots r \qquad (B.6)$$

The $\overline{\Psi}_i$ are annihilators of the R matrix, that is, $R\,\overline{\Psi}_i = 0$, $\overline{\Psi}_i{}'R = 0$. This is easily verified from the definition of R. Since we have assumed that K^* is rank m it follows that R has rank precisely $(m-r)$. The rows of the A_1 matrix are orthogonal to the $\overline{\Psi}_i{}'$, indeed, [illegible] the following

$$\overline{A}_{i1}{}' = i^{\text{th}} \text{ row of } A_1 = \left(|\overline{\Psi}_i|\right)^{-1} \overline{\Psi}_i \qquad i = 1, \cdots r \qquad (B.7)$$

$$i^{\text{th}} \text{ diagonal element of } [A_1 N N A_1{}'] = \left(\frac{\rho_i}{|\overline{\Psi}_i|}\right)^2 . \qquad i = 1, \cdots r \qquad (B.8)$$

Thus $A_1 R = 0$, $\quad R A_1{}' = 0$, and

$$A R A' = \begin{bmatrix} 0 & 0 \\ 0 & (A_2 R A_2{}') \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & (A_2 K^* A_2{}') \end{bmatrix} \qquad (B.9)$$

To establish the last step of equation (B.9) we have used the property

$$A_2 \overline{N}_i^{\nu} = 0 \qquad i = 1, \cdots n \qquad (8.10)$$

which is a consequence of the definition of the A matrix (equation 8.7). Equation (8.9) shows that ≥ 0 implies $(A_1 K^* A_1') \geq 0$ and conversely, which establishes the equivalence to theorem 76.2 of reference 4. An equivalence to the sufficiency condition of theorem 76.3 of reference 4 also can be shown, but the possibility of a singular K^* matrix must then be considered. This analysis will not be presented here.